

So far :

- Quantum systems
 - Quantum states
 - Operations on states
(unitary)
 - Measurements
 - Comp. basis
 - Complete
 - Projective meas.
-

Setup:

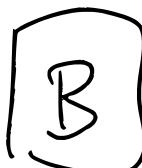
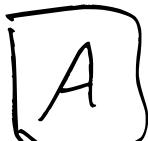
- System has two parts



$$\begin{matrix} 1 & 0 & 0 \\ \downarrow & + & \\ (1) & & (1) \end{matrix}$$

Apply X to part A

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$\cancel{\begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \downarrow & + & & & & \\ (1) & (0) & (1) & (1) & (0) & (1) \end{matrix}}$$

Need to be able to talk about composed systems!

Composing systems (tensor product)

Given system A

$$\{R, G, B\}$$

with N class. poss.

Given system B

with M class. poss.

$$\{\text{up, down}\}$$

What is system AB?

- Has $N \times M$ possibilities

$$(\{R, G, B\} \times \{\text{up, down}\})$$

$$= \{(R, \text{up}), (R, \text{down}), (G, \text{up}), \dots \\ \dots (B, \text{down})\}$$

$$\mathbb{C}^{NM} = \mathbb{C}^N \otimes \mathbb{C}^M$$

(Tensor product of Hilbert spaces)

Composing states

Say: A is in state $|4\rangle \in \mathbb{C}^N$
 B is in state $|4\rangle \in \mathbb{C}^M$

What is the state of AB?

- it is in $\mathbb{C}^{NM} = \mathbb{C}^N \otimes \mathbb{C}^M$

Say: $|4\rangle = |R\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|4\rangle = |\text{down}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then $|4\rangle \otimes |4\rangle = |R, \text{down}\rangle$

For class. poss. i, j:

$$|i\rangle \otimes |j\rangle = |ij\rangle$$

$$\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}_{\leftarrow i-\text{th}} \otimes \begin{pmatrix} 0 \\ j \\ 0 \end{pmatrix}_{\leftarrow j-\text{th}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{\substack{\leftarrow (i,j)-\text{th pos} \\ \text{II} \\ (i \cdot M + j) - \text{th pos}}}$$

General case:

$$|4\rangle = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \quad |4\rangle = \begin{pmatrix} \delta \\ \epsilon \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}_B^R \otimes \begin{pmatrix} \delta \\ \epsilon \\ 0 \end{pmatrix}_{\text{down}}^U = \begin{pmatrix} \alpha\delta \\ \alpha\epsilon \\ \beta\delta \\ \beta\epsilon \\ 0\delta \\ 0\epsilon \end{pmatrix}_{\substack{\text{R-P} \\ \text{Rdown} \\ \text{G-P} \\ \text{Gdown} \\ \text{B-P} \\ \text{Bdown}}} = \begin{pmatrix} \alpha|4\rangle \\ \beta|4\rangle \\ 0|4\rangle \end{pmatrix}$$

Example

Variable X in state $|-\rangle$

Variable Y in state $|0\rangle$

What is XY ?

$$\textcircled{1} \quad |-\rangle \otimes |0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\textcircled{2} \quad |-\rangle \otimes |0\rangle$$

$$= \left(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle) - \frac{1}{\sqrt{2}} (|1\rangle \otimes |0\rangle)$$

$$= \frac{1}{\sqrt{2}} \underset{\substack{\parallel \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{01}^{00} \\ 10 \\ 11}}{|00\rangle} - \frac{1}{\sqrt{2}} \underset{\substack{\parallel \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{01}^{00} \\ 10 \\ 11}}{|10\rangle}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{01}^{00} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{01}^{00} \\ 10 \quad 10 \\ 11 \quad 11$$

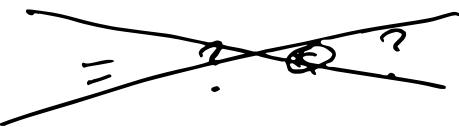
$$= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

In our ^(very) initial example (non-entangled)

$$AB \text{ is in state } |0\rangle \otimes |1\rangle$$
$$= |01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In our entangled setup:

$$AB \text{ is in state } \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle) + \frac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle)$$



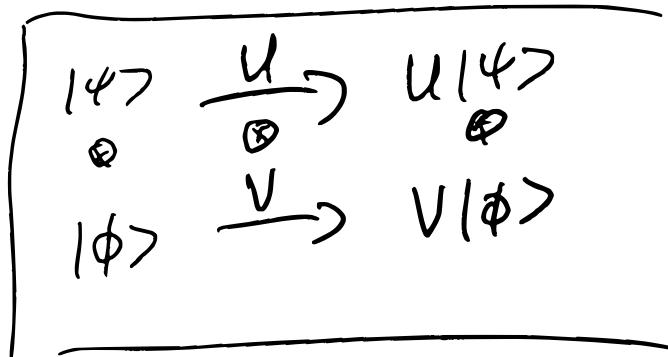
Given a system AB,

how to apply U on A?

... how to apply U on A and
V on B?

If AB is in state $|4\rangle \otimes |\phi\rangle$
then applying $U \otimes V$ to it
leads to:

$$U|4\rangle \otimes V|\phi\rangle$$



First rule: $(U \otimes V)(|4\rangle \otimes |\phi\rangle)$
 $= U|4\rangle \otimes V|\phi\rangle$

Second rule: $U \otimes V$ linear

Why does this determine $U \otimes V$
fully?

Say AB is in state $|r\rangle \in \mathbb{C}^N$

Then $|r\rangle = \sum_{i,j} \alpha_{ij} |ij\rangle$

(eg: our example state

$$\frac{1}{\sqrt{2}} |100\rangle + \frac{1}{\sqrt{2}} |111\rangle$$

$$(U \otimes V) |r\rangle = (U \otimes V) \sum_{i,j} \alpha_{ij} |ij\rangle$$

$$= \sum_{i,j} \alpha_{ij} (U \otimes V) |ij\rangle$$

$$= \sum_{i,j} \alpha_{ij} (U \otimes V) (|i\rangle \otimes |j\rangle)$$

$$= \sum_i \alpha_i (U|i\rangle \otimes V|j\rangle)$$

$U \otimes V$ as a matrix:

$$\begin{pmatrix} u_{11} & \cdots & u_{1N} \\ \vdots & \ddots & \vdots \\ u_{N1} & \cdots & u_{NN} \end{pmatrix} \otimes \begin{pmatrix} v_{11} & \cdots & v_{1M} \\ \vdots & \ddots & \vdots \\ v_{M1} & \cdots & v_{MM} \end{pmatrix} =$$

$$= \begin{pmatrix} u_{11}V & \cdots & u_{1N}V \\ \vdots & \ddots & \vdots \\ u_{N1}V & \cdots & u_{NN}V \end{pmatrix} \in \mathbb{C}^{NM \times NM}$$

Back to the example:

Apply X to A ,

apply ~~$\frac{1}{\sqrt{2}}(I+X)$~~ to B

$\hat{=}$ apply $X \otimes I$ to AB

A

B

$\hat{\Sigma}(00) + \frac{1}{\sqrt{2}}|11\rangle$

$X \downarrow \otimes I \downarrow$

$(X \otimes I) \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \right)$

$= \frac{1}{\sqrt{2}} (X \otimes I)|00\rangle + \frac{1}{\sqrt{2}} (X \otimes I)|11\rangle$

$= \frac{1}{\sqrt{2}} (X \otimes I)(|0\rangle \otimes |0\rangle) + \frac{1}{\sqrt{2}} (X \otimes I)(|1\rangle \otimes |1\rangle)$

$= \frac{1}{\sqrt{2}} \left(\cancel{|0\rangle} \otimes I |0\rangle \right) + \frac{1}{\sqrt{2}} \left(\cancel{|1\rangle} \otimes I |1\rangle \right)$

$= \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|01\rangle$

As matrices

$$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot I & 1 \cdot I \\ 1 \cdot I & 0 \cdot I \end{pmatrix} = \begin{pmatrix} 00 & 10 \\ 00 & 01 \\ 10 & 00 \\ 01 & 00 \end{pmatrix} =: T$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \begin{pmatrix} "1\Sigma \\ 0 \\ 0 \\ 1\Sigma \end{pmatrix} =: |\Gamma\rangle$$

Resulting state: $T|\Gamma\rangle = \begin{pmatrix} 0 \\ "1\Sigma \\ "1\Sigma \\ 0 \end{pmatrix}$

$$? = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$
$$\begin{matrix} " \\ | \\ 0 \\ 1 \\ 0 \end{matrix} \quad \begin{matrix} " \\ | \\ 0 \\ 0 \\ 1 \end{matrix}$$

Measurements (projective)

Example: AB in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
 $= |\beta_{00}\rangle$

We measure A in comp. basis

What happens?

Say we measure A with measurement $M = \{P_1, \dots, P_n\}$

And ignore B

(Meaning we apply meas.
 $I = \{I\}$)

For outcome i,

projector is $P_i \otimes I$

So combined measurement

$$M \otimes I = \{Q_1, \dots, Q_n\}$$

$$\text{where } Q_i = P_i \otimes I$$

Example: AB in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
 $= |\beta_{00}\rangle$

Measure A with :

$$M = \{10 \times 01, 11 \times 11\}$$

Proj. onto $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\stackrel{1}{=} \text{Measure } AB \text{ with } M \otimes I$

$$= \{ |0\rangle\langle 0| \otimes I, |1\rangle\langle 1| \otimes I \}$$

$R\{|0\rangle\langle 0|\}$

$$= \| (|0\rangle\langle 0| \otimes I) |\beta_{00}\rangle \| ^2$$

$$= \left\| \frac{1}{\sqrt{2}} |00\rangle \right\|^2$$

$$= \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

$R\{|1\rangle\langle 1|\}$

$$= \| (|1\rangle\langle 1| \otimes I) |\beta_{00}\rangle \| ^2$$

$$= \left\| \frac{1}{\sqrt{2}} |11\rangle \right\|^2 = \frac{1}{2}$$

p-m-s for 0:

$$\frac{1}{\sqrt{2}} |00\rangle$$

$$\| - \| - \|$$

$$= \frac{1}{\sqrt{2}} |00\rangle$$

$$\frac{1}{\sqrt{2}}$$

$$= |00\rangle = |0\rangle \otimes |0\rangle$$

$$(|0\rangle\langle 0| \otimes I) |\beta_{00}\rangle$$

$$= (|0\rangle\langle 0| \otimes I) \left(\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \right)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle\langle 0| \otimes I) (|00\rangle \otimes |00\rangle)$$

$$+ \frac{1}{\sqrt{2}} (|0\rangle\langle 0| \otimes I) (|11\rangle \otimes |11\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle)$$

$$+ \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle)$$

$$= \frac{1}{\sqrt{2}} |00\rangle$$

p-m-s for 1:

$$|1\rangle \otimes |1\rangle$$

$$= |11\rangle$$