So far:

- Quantum systems
- Quantum states
- Operations on states (unitary)
- Measurements
  - Comp. basis
  - Complete
  - Projective meas.

Setup:

- System has two parts

$\boxed{A}$ $\boxed{B}$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\qquad$ $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Apply $X = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ to part A

$\boxed{A}$ $\boxed{B}$

$X|0\rangle = |1\rangle$ $\qquad$ $|1\rangle$

$\cancel{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}$

Need to be able to talk about composed systems!

Composing systems (tensor product)

Given system A with $N \xrightarrow{} \{R, G, B\}$ class. poss.

Given system B with $M \xrightarrow{} \{up, down\}$ class. poss.

What is system AB?

- Has $N \times M$ possibilities

  ($\{R, G, B\} \times \{up, down\}$

  $= \{(R, up), (R, down), (G, up), \ldots$

  $\ldots, (B, down)\}$)

$$\mathbb{C}^{NM} = \mathbb{C}^{N} \otimes \mathbb{C}^{M}$$

(Tensor product of Hilbert spaces)

Composing states

Say: A is in state $|\psi\rangle \in \mathbb{C}^N$
B is in state $|\phi\rangle \in \mathbb{C}^M$

What is the state of AB?
– it is in $\mathbb{C}^{NM} = \mathbb{C}^N \otimes \mathbb{C}^M$

Say: $|\psi\rangle = |R\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|\phi\rangle = |down\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then $|\psi\rangle \otimes |\phi\rangle = |R, down\rangle$

For class. poss. i, j:

$|i\rangle \otimes |j\rangle = |ij\rangle$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i\text{-th} \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow (i,j)\text{-th pos} = (i \cdot M + j)\text{-th pos}$$

General case:

$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ $|\phi\rangle = \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix}$

$$\underbrace{\begin{array}{cl} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} & \begin{array}{l} R \\ G \\ B \end{array} \end{array}}_{|\psi\rangle} \otimes \underbrace{\begin{array}{cl} \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix} & \begin{array}{l} up \\ down \end{array} \end{array}}_{|\phi\rangle} = \begin{array}{cl} \begin{pmatrix} \alpha\delta \\ \alpha\varepsilon \\ \beta\delta \\ \beta\varepsilon \\ \gamma\delta \\ \gamma\varepsilon \end{pmatrix} & \begin{array}{l} R\,up \\ R\,down \\ G\,up \\ G\,down \\ B\,up \\ B\,down \end{array} \end{array} = \begin{pmatrix} \alpha|\phi\rangle \\ \beta|\phi\rangle \\ \gamma|\phi\rangle \end{pmatrix}$$

Example

Variable X in state $|-\rangle$

Variable Y in state $|0\rangle$

What is XY?

① $|-\rangle \otimes |0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

② $|-\rangle \otimes |0\rangle$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle\right) - \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

$$\underbrace{|00\rangle}_{= \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}} \qquad \underbrace{|10\rangle}_{= \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

In our (very) initial example (non-entangled)

AB is in state $|0\rangle \otimes |1\rangle$
$$= |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

In our entangled setup:

AB is in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
$$= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle\right) + \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |1\rangle\right)$$

$$\cancel{= \ ? \otimes ?}$$

Given a system AB, how to apply U on A?

... how to apply U on A and V on B?

If AB is in state $|\psi\rangle \otimes |\phi\rangle$ then applying $U \otimes V$ to it leads to:

$$U|\psi\rangle \otimes V|\phi\rangle$$

$$\begin{tikzcd}
|\psi\rangle \arrow[r, "U"] & U|\psi\rangle \\[-1.5em]
\otimes \arrow[r, phantom, "\otimes"] & \otimes \\[-1.5em]
|\phi\rangle \arrow[r, "V"] & V|\phi\rangle
\end{tikzcd}$$

First rule: $(U \otimes V)(|\psi\rangle \otimes |\phi\rangle)$
$= U|\psi\rangle \otimes V|\phi\rangle$

Second rule: $U \otimes V$ linear

Why does this determine $U \otimes V$ fully?

Say AB is in state $|\Gamma\rangle \in \mathbb{C}^{NM}$

Then $|\Gamma\rangle = \sum_{ij} \alpha_{ij} |ij\rangle$

(eg: our example state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$)

$$(U \otimes V)|\Gamma\rangle = (U \otimes V) \sum_{ij} \alpha_{ij} |ij\rangle$$

$$= \sum \alpha_{ij} \, (U \otimes V)|ij\rangle$$

$$= \sum \alpha_{ij} \, (U \otimes V)(|i\rangle \otimes |j\rangle)$$

$$= \sum \alpha_{i} \, U|i\rangle \otimes V|j\rangle$$

<u>$U \otimes V$ as a matrix:</u>

$$\underbrace{\begin{pmatrix} u_{11} & \cdots & u_{1N} \\ & & \\ u_{N1} & & u_{NN} \end{pmatrix}}_{= U} \otimes \begin{pmatrix} v_{11} & \cdots & v_{1M} \\ & & \\ v_{M1} & & v_{MM} \end{pmatrix} =$$

$$= \begin{pmatrix} u_{11} V & \cdots & u_{1N} V \\ \vdots & & \vdots \\ u_{N1} V & \cdots & u_{NN} V \end{pmatrix} \in \mathbb{C}^{NM \times NM}$$

Back to the example:

Apply X to A,

apply ~~nothing~~ I to B

$\hat{=}$ apply $X \otimes I$ to AB

$$\begin{array}{ccc} \boxed{A} & & \boxed{B} \\ X \Big\downarrow & \otimes & I \Big\downarrow \end{array} \qquad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$(X \otimes I)\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}(X \otimes I)|00\rangle + \frac{1}{\sqrt{2}}(X \otimes I)|11\rangle$$

$$= \frac{1}{\sqrt{2}}(X \otimes I)(|0\rangle \otimes |0\rangle) + \frac{1}{\sqrt{2}}(X \otimes I)(|1\rangle \otimes |1\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\overset{|1\rangle}{\cancel{X|0\rangle}} \otimes I|0\rangle\right) + \frac{1}{\sqrt{2}}\left(\overset{|0\rangle}{\cancel{X|1\rangle}} \otimes I|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

As matrices

$$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot I & 1 \cdot I \\ 1 \cdot I & 0 \cdot I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =: T$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} =: |\Gamma\rangle$$

Resulting state: $T|\Gamma\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$$\overset{?}{=} \frac{1}{\sqrt{2}} \underset{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}{\underset{\shortparallel}{|01\rangle}} + \frac{1}{\sqrt{2}} \underset{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}{\underset{\shortparallel}{|10\rangle}}$$

Measurements (projective)

Example: AB in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
$= |\beta_{00}\rangle$

We measure A in comp. basis

What happens?

Say we measure A with measurement $\mathcal{M} = \{P_1, \dots, P_n\}$

And ignore B

(Meaning we apply meas. $\mathcal{I} = \{I\}$)

For outcome i, projector is $P_i \otimes I$

So combined measurement

$$\mathcal{M} \otimes \mathcal{I} = \{Q_1, \dots, Q_n\}$$

where $Q_i = P_i \otimes I$

<u>Example</u>: AB in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$= |\beta_{00}\rangle$

Measure A with:

$$\mathcal{M} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

proj. onto $|0\rangle$

$\underline{\underline{1}}$ Measure AB with $\mathcal{M} \otimes I$

$= \{ |0\rangle\langle 0| \otimes I, \ |1\rangle\langle 1| \otimes I \}$

Pr[outcome 0]

$= \| (|0\rangle\langle 0| \otimes I) |\beta_{00}\rangle \|^2$

$= \| \frac{1}{\sqrt{2}} |00\rangle \|^2$

$= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$(|0\rangle\langle 0| \otimes I) |\beta_{00}\rangle$

$= (|0\rangle\langle 0| \otimes I) \left(\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \right)$

$= \frac{1}{\sqrt{2}} (|0\rangle\langle 0| \otimes I)(|0\rangle \otimes |0\rangle)$

$+ \frac{1}{\sqrt{2}} (|0\rangle\langle 0| \otimes I)(|1\rangle \otimes |1\rangle)$

$= \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle)$

$\cancel{+ \frac{1}{\sqrt{2}} (0 \otimes |1\rangle)}$

$= \frac{1}{\sqrt{2}} |00\rangle$

Pr[outcome 1]

$= \| (|1\rangle\langle 1| \otimes I) |\beta_{00}\rangle \|^2$

$= \| \frac{1}{\sqrt{2}} |11\rangle \|^2 = \frac{1}{2}$

p-m-s for 0:

$\dfrac{\frac{1}{\sqrt{2}} |00\rangle}{\| - \text{"} - \|}$

$= \dfrac{\frac{1}{\sqrt{2}} |00\rangle}{1/\sqrt{2}} = |00\rangle = |0\rangle \otimes |0\rangle$

p-m-s for 1:

$|1\rangle \otimes |1\rangle$

$= |11\rangle$